The Complex Mode Spectrum, and Conjugate Mode Pairs, for Rectangular Waveguides Filled with Biaxially Anisotropic Magneto-dielectric

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***Abstract*— We report that an exact analytical solution for the modal electromagnetic fields in a conducting rectangular waveguide filled homogeneously with an un-rotated biaxially anisotropic magneto-dielectric material can be obtained directly by separation of variables, without recourse to numerical methods. It is found that higher order modes are generally hybrid non-TE/TM type modes. It is shown for strongly anisotropic media that there exist pseudo-evanescent complex conjugate mode pairs. This is the first time complex conjugate mode pairs have been directly demonstrated analytically in waveguide completely filled with magneto-dielectric material. Lastly, it is demonstrated that no alternative mode sets exist. The analytical solution given in this paper has not yet been reported in the literature, and could prove useful in guiding efforts to develop Nicolson-Ross type measurements of non-ideal anisotropic behavior exhibited by passive, micro-structured, magneto-dielectric materials. The solution also adds to the portfolio of exact analytical solutions in applied electromagnetic theory, which can be utilized to test the veracity of general purpose numerical codes.**

***Index Terms*— Complex modes, anisotropic materials, magnetic materials, electromagnetic theory.**

I. INTRODUCTION

In this report, an exact analytic solution is given for the electromagnetic modes supported by a conducting rectangular waveguide filled homogeneously with a composite magneto-dielectric material described by permittivity and permeability tensors. Such materials can be realized in several ways: by structuring of multiple different material phases [1-3]; by treatment of soft bulk ceramic polycrystalline ferrites in strong magnetic fields during crystallization [4]; by formation of bulk ferrites in the form of a single large crystal [4]; or by constructing thin-film composites with preferentially aligned single-domain ferromagnetic metallic alloy flakes, strips, or rods [5-6]. In developing such composite materials, the crystal structure or orientation of the inclusions is controlled to enhance the magnetic or electric properties of the material, but often with the side effect of making the material's properties directionally dependent, or anisotropic.

The mathematical boundary value problem for modeling the magneto-dielectric-waveguide system was considered by Meng et al [7]. We note that in [7] an analytical solution is presented. However, their formulas are only valid for the TE_{0n} and TE_{m0} modes. The formulation presented in [7] yields field solutions, that for all modes beside the TE_{0n} and TE_{m0} modes, do not satisfy Maxwell's equations. Hence, their formulation does not provide accurate expressions for the remaining higher order modes. The subsequent numerical analysis in [7] was only applied in the special case of uniaxial anisotropy, and in this case, their field solutions reduce to forms that do correctly satisfy Maxwell's equations.

Here, we give a derivation of the complete mode set for the case of biaxial anisotropy, including the remaining higher order modes. Such theoretical information is potentially of importance for developing Nicolson-Ross type measurement procedures for the empirical investigation of the non-ideal anisotropic characteristics of microwave magnetic materials [8].

The mathematical procedure is essentially separation of variables. However, the anisotropy of the magneto-dielectric filling necessitates the solution of an eigenvalue-eigenvector problem for the propagation constants and field components of the individual modes, which adds considerably to the complexity and richness of the observant electromagnetic phenomena.

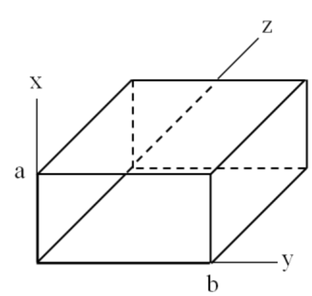

Fig. 1. Geometry of the waveguide.

II. The Rectangular Waveguide Filled with Biaxial Magneto-Dielectric: separation of variables solution

We presume that the walls of the rectangular waveguide are perfectly conducting. The geometry of the waveguide is illustrated in Fig.1. The waveguide is filled homogeneously with a biaxially anisotropic magneto-dielectric material aligned with the Cartesian coordinate axes such that it can be described by the constitutive tensors

$$\bar{\boldsymbol{\varepsilon}} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} , \quad (1)$$

$$\bar{\boldsymbol{\mu}} = \begin{bmatrix} \mu_x & 0 & 0 \\ 0 & \mu_y & 0 \\ 0 & 0 & \mu_z \end{bmatrix} , \quad (2)$$

Which, assuming $e^{j\omega t}$ time dependence, enter into Maxwell's curl equations as follows:

$$\nabla \times \boldsymbol{E} = -j\omega\mu_0 \bar{\boldsymbol{\mu}} \cdot \boldsymbol{H} \; ; \quad (3)$$

$$\nabla \times \boldsymbol{H} = j\omega\varepsilon_0 \bar{\boldsymbol{\varepsilon}} \cdot \boldsymbol{E} \; . \quad (4)$$

It is convenient to represent the tensor equations (3) and (4) explicitly in terms of their scalar components in Cartesian coordinates,

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu_0\mu_x H_x \; , \quad (5)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu_0\mu_y H_y \; , \quad (6)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu_0\mu_z H_z \; , \quad (7)$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon_0\varepsilon_x E_x \; , \quad (8)$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon_0\varepsilon_y E_y \; , \quad (9)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon_0\varepsilon_z E_z \; . \quad (10)$$

In a source free region, all six field components are not independent, and any two may be eliminated. We will proceed by solving (7) and (10) for H_z, E_z, and then substituting into the remaining equations to obtain four coupled partial differential equations involving the transverse field components,

$$\frac{\partial E_x}{\partial z} = -\frac{j\eta_0}{\varepsilon_z k_0}\left[\mu_y\varepsilon_z {k_0}^2 H_y + \frac{\partial^2 H_y}{\partial x^2} - \frac{\partial^2 H_x}{\partial x \partial y}\right] , \quad (11)$$

$$\frac{\partial E_y}{\partial z} = \frac{j\eta_0}{\varepsilon_z k_0}\left[\mu_x\varepsilon_z {k_0}^2 H_x + \frac{\partial^2 H_x}{\partial y^2} - \frac{\partial^2 H_y}{\partial x \partial y}\right] , \quad (12)$$

$$\frac{\partial H_x}{\partial z} = \frac{j}{\mu_z\eta_0 k_0}\left[\varepsilon_y\mu_z {k_0}^2 E_y + \frac{\partial^2 E_y}{\partial x^2} - \frac{\partial^2 E_x}{\partial x \partial y}\right] , \quad (13)$$

$$\frac{\partial H_y}{\partial z} = -\frac{j}{\mu_z\eta_0 k_0}\left[\varepsilon_x\mu_z {k_0}^2 E_x + \frac{\partial^2 E_x}{\partial y^2} - \frac{\partial^2 E_y}{\partial x \partial y}\right] , \quad (14)$$

where $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ and $k_0 = \omega\sqrt{\mu_0\varepsilon_0}$. We define the state vector for the electromagnetic system to be the column vector given by:

$$\vec{\boldsymbol{\psi}} = [E_x \quad E_y \quad H_x \quad H_y]^T \; . \quad (15)$$

Inspection of (11)-(14) enables us to reformulate Maxwell's equations as the linear operator equation, given in state-space representation as:

$$\frac{\partial \vec{\boldsymbol{\psi}}}{\partial z} = L(\vec{\boldsymbol{\psi}}) \; , \quad (16)$$

where L is the linear matrix operator

$$L \to \begin{pmatrix} 0 & 0 & L_{13} & L_{14} \\ 0 & 0 & L_{23} & L_{24} \\ L_{31} & L_{32} & 0 & 0 \\ L_{41} & L_{42} & 0 & 0 \end{pmatrix} , \quad (17)$$

with elements

$$L_{13} \to \frac{j\eta_0}{\varepsilon_z k_0}\frac{\partial^2}{\partial x \partial y} \; , \quad (18)$$

$$L_{14} \to -\frac{j\eta_0}{\varepsilon_z k_0}\left(\mu_y\varepsilon_z {k_0}^2 + \frac{\partial^2}{\partial x^2}\right) , \quad (19)$$

$$L_{23} \to \frac{j\eta_0}{\varepsilon_z k_0}\left(\mu_x\varepsilon_z {k_0}^2 + \frac{\partial^2}{\partial y^2}\right) , \quad (20)$$

$$L_{24} \to -\frac{j\eta_0}{\varepsilon_z k_0}\frac{\partial^2}{\partial x \partial y} \; , \quad (21)$$

$$L_{31} \to -\frac{j}{\mu_z\eta_0 k_0}\frac{\partial^2}{\partial x \partial y} \; , \quad (22)$$

$$L_{32} \to \frac{j}{\mu_z\eta_0 k_0}\left(\varepsilon_y\mu_z {k_0}^2 + \frac{\partial^2}{\partial x^2}\right) , \quad (23)$$

$$L_{41} \to -\frac{j}{\mu_z\eta_0 k_0}\left(\varepsilon_x\mu_z {k_0}^2 + \frac{\partial^2}{\partial y^2}\right) , \quad (24)$$

$$L_{42} \to \frac{j}{\mu_z\eta_0 k_0}\frac{\partial^2}{\partial x \partial y} \; . \quad (25)$$

We presume variables can be separated, and suppose that the modal fields take the form

$$E_x = \widetilde{E_x}\cos(k_x x)\sin(k_y y) \; , \quad (26)$$

$$E_y = \widetilde{E_y}\sin(k_x x)\cos(k_y y) \; , \quad (27)$$

$$H_x = \widetilde{H_x}\sin(k_x x)\cos(k_y y) \; , \quad (28)$$

$$H_y = \widetilde{H_y}\cos(k_x x)\sin(k_y y) \; , \quad (29)$$

where, for any positive whole numbers m and n, $k_x = m\pi/a$ and $k_y = n\pi/b$. The particular choices of cosine or sine in these expressions are made to satisfy the boundary conditions at the conducting walls of the waveguide. We refer to $\widetilde{E_x}, \widetilde{E_y}, \widetilde{H_x}$, and $\widetilde{H_y}$ as the spectral fields, which physically represent the weighting coefficients for the individual modes, and we define the spectral electromagnetic state vector to be the column vector

$$\widetilde{\widetilde{\boldsymbol{\psi}}} = \left[\widetilde{E_x} \quad \widetilde{E_y} \quad \widetilde{H_x} \quad \widetilde{H_y}\right]^T \ . \tag{30}$$

By substitution of (26-29) into (16), the spectral state vector can be shown to satisfy the ordinary matrix differential equation,

$$\frac{d\widetilde{\widetilde{\boldsymbol{\psi}}}}{dz} = \bar{\bar{\boldsymbol{\Gamma}}} \cdot \widetilde{\widetilde{\boldsymbol{\psi}}} \ , \tag{31}$$

where $\bar{\bar{\boldsymbol{\Gamma}}}$ is the 4x4 matrix

$$\bar{\bar{\boldsymbol{\Gamma}}} = \begin{bmatrix} \bar{\boldsymbol{0}} & \bar{\boldsymbol{Z}} \\ \bar{\boldsymbol{Y}} & \bar{\boldsymbol{0}} \end{bmatrix} \ , \tag{32}$$

and where we have also defined 2x2 impedance and admittance matrices

$$\bar{\boldsymbol{Z}} = \frac{j\eta_0}{\varepsilon_z k_0} \begin{bmatrix} -k_x k_y & {k_x}^2 - \mu_y \varepsilon_z {k_0}^2 \\ \mu_x \varepsilon_z {k_0}^2 - {k_y}^2 & k_x k_y \end{bmatrix} \ , \tag{33}$$

$$\bar{\boldsymbol{Y}} = \frac{j}{\mu_z \eta_0 k_0} \begin{bmatrix} k_x k_y & \varepsilon_y \mu_z {k_0}^2 - {k_x}^2 \\ {k_y}^2 - \varepsilon_x \mu_z {k_0}^2 & -k_x k_y \end{bmatrix} \ , \tag{34}$$

and $\bar{\boldsymbol{0}}$ as the 2x2 matrix of zeroes. Further, we define column vectors for the spectral transverse electric and magnetic fields:

$$\widetilde{\widetilde{\overrightarrow{\boldsymbol{E_t}}}} = \left[\widetilde{E_x} \quad \widetilde{E_y}\right]^T \ ; \tag{35}$$

$$\widetilde{\widetilde{\overrightarrow{\boldsymbol{H_t}}}} = \left[\widetilde{H_x} \quad \widetilde{H_y}\right]^T \ . \tag{36}$$

Then, (31) splits into the two separate matrix differential equations,

$$\frac{d\widetilde{\widetilde{\overrightarrow{\boldsymbol{E_t}}}}}{dz} = \bar{\boldsymbol{Z}} \cdot \widetilde{\widetilde{\overrightarrow{\boldsymbol{H_t}}}} \ , \tag{37}$$

$$\frac{d\widetilde{\widetilde{\overrightarrow{\boldsymbol{H_t}}}}}{dz} = \bar{\boldsymbol{Y}} \cdot \widetilde{\widetilde{\overrightarrow{\boldsymbol{E_t}}}} \ . \tag{38}$$

Taking the derivative of (37), we obtain

$$\frac{d^2\widetilde{\widetilde{\overrightarrow{\boldsymbol{E_t}}}}}{dz^2} = \bar{\boldsymbol{Z}} \cdot \frac{d\widetilde{\widetilde{\overrightarrow{\boldsymbol{H_t}}}}}{dz} \ . \tag{39}$$

Substitution of (38) into (39) yields

$$\frac{d^2\widetilde{\widetilde{\overrightarrow{\boldsymbol{E_t}}}}}{dz^2} = \bar{\boldsymbol{Z}} \cdot \bar{\boldsymbol{Y}} \cdot \widetilde{\widetilde{\overrightarrow{\boldsymbol{E_t}}}} \ . \tag{40}$$

Equation (40) is a two-equation system of second order differential equations. Substitution of a trial solution of the form $e^{\pm\sqrt{\lambda}z}\vec{\boldsymbol{v}}$ into (40) gives

$$\frac{d^2}{dz^2}\left(e^{\pm\sqrt{\lambda}z}\vec{\boldsymbol{v}}\right) = \bar{\boldsymbol{Z}} \cdot \bar{\boldsymbol{Y}} \cdot \left(e^{\pm\sqrt{\lambda}z}\vec{\boldsymbol{v}}\right) \ ,$$

$$e^{\pm\sqrt{\lambda}z}(\bar{\boldsymbol{Z}} \cdot \bar{\boldsymbol{Y}} \cdot \vec{\boldsymbol{v}} - \lambda\vec{\boldsymbol{v}}) = 0 \ ,$$

$$\bar{\boldsymbol{Z}} \cdot \bar{\boldsymbol{Y}} \cdot \vec{\boldsymbol{v}} = \lambda\vec{\boldsymbol{v}} \ . \tag{41}$$

Equation (41) indicates that $e^{\pm\sqrt{\lambda}z}\vec{\boldsymbol{v}}$ are two independent solutions of (40) provided that $\vec{\boldsymbol{v}}, \lambda$ are an eigenvector and associated eigenvalue, respectively, of the matrix $\bar{\boldsymbol{Z}} \cdot \bar{\boldsymbol{Y}}$. Since $\bar{\boldsymbol{Z}} \cdot \bar{\boldsymbol{Y}}$ has two eigenvectors, we have four total solutions to (40). It follows that $\pm\sqrt{\lambda_1}$, $\pm\sqrt{\lambda_2}$ are the four eigenvalues of the matrix $\bar{\bar{\boldsymbol{\Gamma}}}$. For the remainder of this paper, when we say eigenvalue, we unambiguously mean either λ_1 or λ_2. We write the general solution of (40) as,

$$\widetilde{\widetilde{\overrightarrow{\boldsymbol{E_t}}}} = \widetilde{c_1}e^{-\sqrt{\lambda_1}}\overrightarrow{\boldsymbol{v_1}} + \widetilde{c_2}e^{\sqrt{\lambda_1}}\overrightarrow{\boldsymbol{v_1}} + \widetilde{c_3}e^{-\sqrt{\lambda_2}}\overrightarrow{\boldsymbol{v_2}} + \widetilde{c_4}e^{\sqrt{\lambda_2}}\overrightarrow{\boldsymbol{v_2}} \ , \tag{42}$$

where $\widetilde{c_1}$-$\widetilde{c_4}$ are arbitrary constants, and $\overrightarrow{\boldsymbol{v_1}}, \overrightarrow{\boldsymbol{v_2}}$ are the two eigenvectors of the matrix $\bar{\boldsymbol{Z}} \cdot \bar{\boldsymbol{Y}}$. The transversal spectral magnetic field may be obtained using (37) and (42) as,

$$\widetilde{\widetilde{\overrightarrow{\boldsymbol{H_t}}}} = -\widetilde{c_1}\sqrt{\lambda_1}e^{-\sqrt{\lambda_1}}\bar{\boldsymbol{Z}}^{-1} \cdot \overrightarrow{\boldsymbol{v_1}} + \widetilde{c_2}\sqrt{\lambda_1}e^{\sqrt{\lambda_1}}\bar{\boldsymbol{Z}}^{-1} \cdot \overrightarrow{\boldsymbol{v_1}} - \widetilde{c_3}\sqrt{\lambda_2}e^{-\sqrt{\lambda_2}}\bar{\boldsymbol{Z}}^{-1} \cdot \overrightarrow{\boldsymbol{v_2}} + \widetilde{c_4}\sqrt{\lambda_2}e^{\sqrt{\lambda_2}}\bar{\boldsymbol{Z}}^{-1} \cdot \overrightarrow{\boldsymbol{v_2}} \ . \tag{43}$$

Equations (42) and (43), taken together, form the general solution to the original matrix equation (31).

$$\widetilde{\widetilde{\boldsymbol{\psi}}} = \widetilde{c_1}e^{-\sqrt{\lambda_1}z}\begin{bmatrix} \overrightarrow{\boldsymbol{v_1}} \\ -\sqrt{\lambda_1}\bar{\bar{\boldsymbol{Z}}}^{-1} \cdot \overrightarrow{\boldsymbol{v_1}} \end{bmatrix} + \widetilde{c_2}e^{\sqrt{\lambda_1}z}\begin{bmatrix} \overrightarrow{\boldsymbol{v_1}} \\ \sqrt{\lambda_1}\bar{\bar{\boldsymbol{Z}}}^{-1} \cdot \overrightarrow{\boldsymbol{v_1}} \end{bmatrix} + \widetilde{c_3}e^{-\sqrt{\lambda_2}z}\begin{bmatrix} \overrightarrow{\boldsymbol{v_2}} \\ -\sqrt{\lambda_2}\bar{\bar{\boldsymbol{Z}}}^{-1} \cdot \overrightarrow{\boldsymbol{v_2}} \end{bmatrix} + \widetilde{c_4}e^{\sqrt{\lambda_2}z}\begin{bmatrix} \overrightarrow{\boldsymbol{v_2}} \\ \sqrt{\lambda_2}\bar{\bar{\boldsymbol{Z}}}^{-1} \cdot \overrightarrow{\boldsymbol{v_2}} \end{bmatrix} \tag{44}$$

Equation (44) may then be combined with (26)-(29) to synthesize the field expressions for the complete mode spectrum supported by the waveguide filled homogeneously with biaxially anisotropic magneto-dielectric. The axial fields, if desired, may be obtained from (7) and (10).

We note that (44) indicates two mode types, one propagates with propagation constant $\sqrt{\lambda_1}$, the other $\sqrt{\lambda_2}$, and the field components exist in different proportions according to the form of the two different eigenvectors. This result is at variance with equations (12) and (19) of [7], which assign two propagation constants to a single mode type. A single mode type can have only one propagation constant. This is true even for birefringent crystals, where birefringence results from coupling to two separate modes, each with different propagation constants, not coupling to a single mode with two different propagation constants.

The modes of both types, of various orders m and n, are obtained from (26)-(29). Equation (44) shows that the waveguide supports propagation of both mode types in either direction, verifying the expectation that mode propagation is reciprocal. That is, the four solutions to (31) consist of the two mode types, forward and backward propagating. This is known not to be true for magnetic materials exhibiting a permanent or forced DC magnetization [9]. We could have also concluded immediately that propagation would be reciprocal since $\bar{\boldsymbol{\mu}} = \bar{\boldsymbol{\mu}}^T$ and $\bar{\boldsymbol{\varepsilon}} = \bar{\boldsymbol{\varepsilon}}^T$, which is a sufficient condition for the Lorentz reciprocity theorem to hold [10].

III. THE PROPAGATION CONSTANTS AND EXISTENCE OF COMPLEX CONJUGATE MODE PAIRS

The solution to the eigenvalue equation (41) may be obtained by techniques of linear algebra. For the 2x2 eigenvalue-eigenvector system, explicit formulas exist for the eigenvalues and eigenvectors. Applying these formulas leads to the following solutions. We obtain for the eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\Big[\Big(\frac{\varepsilon_x}{\varepsilon_z} + \frac{\mu_x}{\mu_z}\Big) k_x{}^2 + \Big(\frac{\varepsilon_y}{\varepsilon_z} + \frac{\mu_y}{\mu_z}\Big) k_y{}^2 - \big(\varepsilon_x\mu_y + \mu_x\varepsilon_y\big)k_0{}^2\Big] \pm \Big\{\frac{1}{4}\Big[\Big(\frac{\varepsilon_x}{\varepsilon_z} - \frac{\mu_x}{\mu_z}\Big) k_x{}^2 + \Big(\frac{\mu_y}{\mu_z} - \frac{\varepsilon_y}{\varepsilon_z}\Big) k_y{}^2 + \big(\mu_x\varepsilon_y - \varepsilon_x\mu_y\big)k_0{}^2\Big]^2 + k_x{}^2 k_y{}^2 \Big(\frac{\varepsilon_x}{\varepsilon_z} - \frac{\mu_x}{\mu_z}\Big)\Big(\frac{\varepsilon_y}{\varepsilon_z} - \frac{\mu_y}{\mu_z}\Big)\Big\}^{1/2} \qquad (45)$$

where $\pm$ indicates the two different eigenvalues $\lambda_{1,2}$ of the matrix $\bar{\boldsymbol{Z}} \cdot \bar{\boldsymbol{Y}}$, corresponding to the two different mode types. A single formula cannot be given for the eigenvectors, and a number of special cases must be enumerated. All special cases can be derived by solving (41) subject to (45).

Case I: $k_x \neq 0, k_y \neq 0, \varepsilon_y\mu_z \neq \varepsilon_z\mu_y$:

$$\overrightarrow{\boldsymbol{v}_{1,2}} = \begin{bmatrix} k_x k_y \Big(\frac{\varepsilon_y}{\varepsilon_z} - \frac{\mu_y}{\mu_z}\Big) \\ \lambda_{1,2} - \frac{\varepsilon_x}{\varepsilon_z} k_x{}^2 - \frac{\mu_y}{\mu_z} k_y{}^2 + \varepsilon_x\mu_y k_0{}^2 \end{bmatrix} . \qquad (46)$$

Case II: $k_x \neq 0, k_y \neq 0, \varepsilon_x\mu_z \neq \varepsilon_z\mu_x$:

$$\overrightarrow{\boldsymbol{v}_{1,2}} = \begin{bmatrix} \lambda_{1,2} - \frac{\mu_x}{\mu_z} k_x{}^2 - \frac{\varepsilon_y}{\varepsilon_z} k_y{}^2 + \mu_x\varepsilon_y k_0{}^2 \\ k_x k_y \Big(\frac{\varepsilon_x}{\varepsilon_z} - \frac{\mu_x}{\mu_z}\Big) \end{bmatrix} . \qquad (47)$$

Case III: $k_x \neq 0, k_y \neq 0, \bar{\boldsymbol{\varepsilon}} \propto \bar{\boldsymbol{\mu}}$:

$$\vec{\boldsymbol{v}} = \widetilde{c_1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \widetilde{c_2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad . \qquad (48)$$

Case IV: $k_x = 0, k_y \neq 0$:

$$\overrightarrow{\boldsymbol{v}_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \qquad (49)$$

Case V: $k_x \neq 0, k_y = 0$:

$$\overrightarrow{\boldsymbol{v}_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \qquad (50)$$

There is physical significance to each of these special cases. Cases I and II correspond to general biaxial anisotropy, and (46) and (47) result in the higher order hybrid modes for which the electric and magnetic fields in the direction of propagation are both nonzero. Two cases are needed to accommodate the possibility of partial symmetries between the permittivity and permeability tensors, which zero out the off-diagonal entries in the matrix $\bar{\boldsymbol{Z}} \cdot \bar{\boldsymbol{Y}}$, changing the character of the eigenvalue equation (41). In the case that neither of the partial symmetries occur, the eigenvectors given in equations (46) and (47) are directly proportional, yielding the same solution. The hybrid modes are the modes that we are particularly interested in. We shall summarize the mode behavior for cases III-V before further investigation of these higher order hybrid modes.

Case III is for a very special type of anisotropy where the permittivity and permeability matrices are directly proportional. In this highly contrived scenario, the matrix $\bar{\boldsymbol{Z}} \cdot \bar{\boldsymbol{Y}}$ becomes diagonal. The eigenvalues are repeating, creating a degeneracy requiring two eigenvectors to be assigned to a single eigenvalue. Equation (48) will lead to TE^x/TE^y mode sets, but other classifications could also be made by taking combinations of the degenerate modes. This procedure is discussed in the context of isotropic media in the next section.

Cases IV and V lead to TE_{0n} and TE_{m0} modes respectively, regardless of the form of the permeability or permittivity matrices. They are the only TE/TM modes existing in the general biaxially anisotropic guide. These modes can also be found in the paper by Meng et al, provided that we set $k_x = 0$ or $k_y = 0$, and only retain the particular propagation constant that is found in agreement with our formula [7]. Depending on the dimensions of the waveguide, either the TE_{01} or the TE_{10} mode is dominant in the anisotropic guide, which is no different than in an isotropic rectangular waveguide. For Cases IV and V, only $\overrightarrow{\boldsymbol{v}_1}$ is given, because the fields for the other mode type vanish when either $k_x = 0$ or $k_y = 0$.

We note that another special case occurs when biaxial anisotropy reduces to uniaxial anisotropy, such as when $\mu_x = \mu_y$ and $\varepsilon_x = \varepsilon_y$. This, however, is already contained in cases I-III. For uniaxial anisotropy, all modes are either TE or TM. The last special case is fully isotropic material, which will be discussed in the context of this derivation in the next section.

Equations (45-50) reveal a complicated dependency of the modal fields on the permittivity and permeability tensors. Importantly, it appears from (45) that even if all elements of the permittivity and permeability matrices are real, there is, in general, no single well-defined cut-off frequency for each mode. This can be seen by setting the eigenvalues in (45) equal to 0, and noting that the resulting equation will not always have exactly one positive real root for the square of the cutoff wavenumber ${k_0}^2$. In any situation where there is not exactly one real, positive root of (45) for ${k_0}^2$, it is not possible to define a single cutoff frequency in the usual way.

To illustrate this behavior, we now consider a typical example. For simplicity, the material's properties are taken as non-dispersive. This, of course, is not correct; especially for magnetic materials which exhibit various resonances including domain wall resonances and ferromagnetic resonance [4]. However, taking this shortcut will allow us to demonstrate the strange behavior of the propagation constant. We let: $\varepsilon_x = 2, \varepsilon_y = 4, \varepsilon_z = 15, \mu_x = 8, \mu_y = 3, \mu_z = 17, a = b = .05$ (meters), $m = n = 1$. In Figs. 2 and 3, we let $\sqrt{\lambda_i} = \alpha_i + j\beta_i$, and plot α_i, β_i, respectively, versus frequency f for frequencies ranging from DC up to 2 GHz.

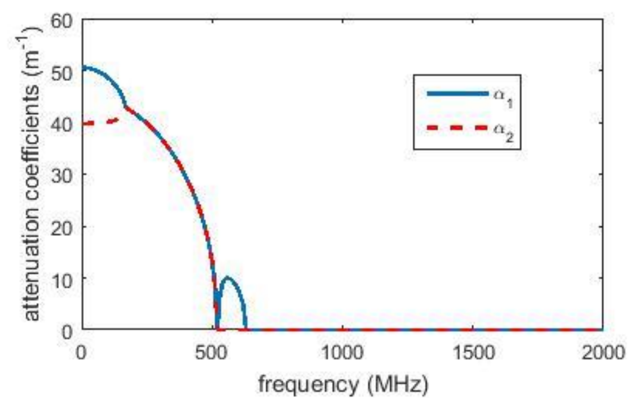

Fig. 2. Attenuation coefficient vs. frequency.

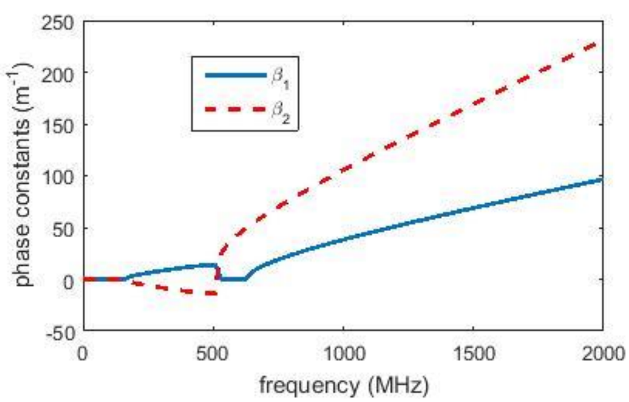

Fig. 3. Wavenumber vs. frequency.

We observe from Figs.2 and 3 that there are no well-defined cutoff frequencies for either mode type. The odd-looking kinks in the plots result from mapping the real and imaginary parts of a complex function of frequency onto separate plots. There are multiple points at which the eigenvalues are equal to 0, and they demarcate transition frequencies at which the behavior of the waveguide transitions. It can be seen that although a single cutoff cannot be defined, there is a minimum frequency at which non-attenuated propagation can occur. Above this frequency, the behavior of the wavenumber is analogous to what would occur in a waveguide filled with isotropic material, and the modes carry power. Below these frequencies, there exists an anomalous dispersion frequency region where complex modes propagate with attenuation, and exhibit backward-wave behavior. In our example, the onset of these complex modes is at approximately 500 MHz. It can be deduced from (45) that in the anomalous frequency region, the complex modes appear as a conjugate pair, which is necessary to ensure that taken as a pair the pseudo-evanescent modes carry no overall time-averaged real power. In our example, it would make little sense for only one of the $m = n = 1$ modes to exist without the other, since it would carry time-average power, but with attenuation, implying a loss of energy without a physical mechanism, as the material parameters need not, in theory, have an imaginary part for this behavior to manifest. For physically realizable fields, it follows that the presence of one such mode must imply the other.

The existence and properties of such complex mode pairs was discussed by Rozzi et. al. for the case of lossless isotropic closed waveguides of arbitrary cross section, and were explained as another reactive mechanism of the waveguide, analogous to a single real mode below cutoff [11]. Indeed, in agreement with the requirement set forth by equation (4) of [11], we note from (43) and (46-47) that the transversal electric fields of the two modes in the anomalous frequency region are the complex conjugate of each other. To the author's knowledge, in the context of an anisotropic waveguide that is closed, lossless, and reciprocal, this phenomenon has not been directly demonstrated before now.

IV. COMPARISON WITH FORMULAS FOR AN ISOTROPIC MATERIAL: NONEXISTENCE OF ALTERNATIVE MODE SETS

It is of interest here to show that our formulas reduce to the correct form when we set $\varepsilon_x = \varepsilon_y = \varepsilon_z$ and $\mu_x = \mu_y = \mu_z$. First, in isotropic media, we note that (45) reduces to

$$\lambda = -{k_z}^2 = {k_x}^2 + {k_y}^2 - k^2 \; . \qquad (51)$$

Equation (51) is recognized as the well-known separation, or dispersion, equation in isotropic media. We drop the 1,2 subscript, because the eigenvalues (propagation constants) for both mode types have the same form in an isotropic waveguide.

We can obtain the eigenvectors in isotropic media from (41), but the general character of the eigenvalue equation is different for repeating eigenvalues. In fact, substitution of (51) into (41), and simplification of the expressions for the matrix $\bar{\boldsymbol{Z}} \cdot \bar{\boldsymbol{Y}}$ yields the following equation, which places essentially no restriction on the eigenvectors:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \vec{\boldsymbol{v}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \qquad (52)$$

This has, among other possible solutions, the trivial solution,

$$\vec{\boldsymbol{v}} = \tilde{c_1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \tilde{c_2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \qquad (53)$$

Equation (53) shows that the eigenspace for the repeated eigenvalue in isotropic media is two-dimensional; spanned by two linearly independent basis vectors. We can rotate to other basis vectors that span this space and obtain alternative mode sets such as those described by Harrington [12]. Rotating the basis vectors within this eigenspace is equivalent to taking combinations of the degenerate modes, that is, modes with the same propagation constant, to form a different set of modes. This can only be done when dealing with isotropic media. We note that in the biaxially anisotropic magneto-dielectric waveguide, excluding case III described above, eigenvalues do not repeat, the propagation constants are unique, and therefore alternative mode sets do not exist.

The usual TE and TM modes in a rectangular waveguide filled with isotropic media may be obtained by setting:

$$\vec{\boldsymbol{v}} = \widetilde{c_1}\begin{bmatrix} k_x k_z \\ k_y k_z \end{bmatrix} + \widetilde{c_2}\begin{bmatrix} \omega\mu k_y \\ -\omega\mu k_x \end{bmatrix}, \tag{54}$$

which, in our general framework, leads to the solution

$$\begin{aligned} \widetilde{\vec{\boldsymbol{\psi}}} = e^{-jk_z z}\left(\widetilde{c_1}\begin{bmatrix} k_x k_z \\ k_y k_z \\ -\omega\varepsilon k_y \\ \omega\varepsilon k_x \end{bmatrix} + \widetilde{c_2}\begin{bmatrix} \omega\mu k_y \\ -\omega\mu k_x \\ k_x k_z \\ k_y k_z \end{bmatrix}\right) \\ + e^{jk_z z}\left(\widetilde{c_3}\begin{bmatrix} k_x k_z \\ k_y k_z \\ \omega\varepsilon k_y \\ -\omega\varepsilon k_x \end{bmatrix} + \widetilde{c_4}\begin{bmatrix} \omega\mu k_y \\ -\omega\mu k_x \\ -k_x k_z \\ -k_y k_z \end{bmatrix}\right). \end{aligned} \tag{55}$$

Usage of (55) in (26)-(29) gives the TE^z/TM^z (transverse to the z direction) modal decomposition for a waveguide filled with isotropic material [12]. We note that if we instead used (53), we would obtain the TE^x/TE^y alternative mode sets described by Harrington [12]. We reiterate that alternative mode sets do not exist for the waveguide filled with anisotropic magneto-dielectric.

V. Conclusions

In this report, a theoretical framework was developed to analyze the rectangular waveguide filled homogeneously with un-rotated biaxial magneto-dielectric material. It was found that the waveguide supported two mode types with reciprocal propagation, which for most of the higher order modes, can be classified as hybrid, as they are neither TM nor TE to any of the Cartesian coordinate axes. The existence of complex conjugate pseudo-evanescent mode pairs, and the non-existence of alternative mode sets, clearly distinguishes the biaxial magneto-dielectric filling from simpler isotropic filling. The results obtained in this paper extend the range of validity of the results given by Meng et al., providing the complete mode spectrum [7]. We note that for more complicated anisotropies, including a magnetic material such as that considered in this paper, but misaligned with the coordinate axes, that an analytic solution cannot be obtained by the same methods. For more complex anisotropies, numerical techniques are required, and the results will take the form of truncated infinite series to approximate the fields and propagation constant of a single mode [13], as opposed to the closed form expressions (44-50). We note that our analytical solution could be utilized to build confidence in these numerical schemes, particularly in their ability to predict the unusual features discussed in this paper that often have no counterpart in problems involving isotropic media, such as the hybrid nature of the higher order modes and the existence of complex conjugate mode pairs. The analytical solution also provides useful theoretical input into development of Nicolson-Ross type measurement techniques for characterization of anisotropic behavior exhibited by certain microwave magnetic materials such as multi-phase magnetic composites, single-crystal ceramic ferrites, and thin-film ferromagnetic composites [1-6, 8].

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