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# Finite Element Analysis of Rectangular Waveguide with Stochastic Geometric Variations

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***Abstract*—The analysis of rectangular PEC-wall waveguides is a classical boundary value problem with well-known analytical solutions. The modal cutoff frequencies and dispersion characteristics of any waveguide geometry can be easily computed using the Finite Element Method. This paper focuses on the analysis of rectangular waveguides with geometric uncertainties. By leveraging coordinate transformation and stochastic collocation, the statistics of the waveguide characteristics can be quickly derived without the need to perform lengthy Monte Carlo analysis. Results are verified against analytical and Monte Carlo solutions.**

***Index Terms*—Stochastic, waveguide, finite element, stochastic collocation, geometric uncertainty**

## I. Introduction

The Finite Element Method (FEM) is a versatile numerical procedure to find approximate solutions to partial differential equations in engineering applications [**?**], [**?**]. Over the past two decades, FEM has been gaining popularity in the computational electromagnetics community as an effective method for solving many types of boundary value problems. Many of today's industry standard electromagnetic solvers are based on FEM.

A major advantage of FEM is its utilization of geometric meshes to discretize the solution domains. This allows problems with various complicated geometries to be solved easily using the same formulation. The shapes and densities of the meshes can also be customized to suit the problems under consideration, so a balance between accuracy and speed can be achieved.

This chapter studies the stochastic dispersion characteristics of rectangular waveguides with geometric uncertainties. Rectangular waveguide problems can be easily formulated as closed-domain boundary value problems using FEM. The various propagation modes of a given waveguide correspond to the eigenvalues and eigenvectors of the resulting set of algebraic equations from FEM discretization. Although the deterministic solutions to rectangular waveguide problems can be found analytically, the solutions for random cross-sections will be more involved. This problem is interesting to consider since we expect waveguides that are manufactured to exhibit some uncertainty in its shape due to variations in the manufacturing process. With uncertainty quantifications of the geometry, it is useful to perform classical waveguide analysis to obtain characteristics such as modal cutoff frequencies and dispersion diagrams as random variables, so we can understand the probabilistic distributions of these parameters.

FEM is the ideal method to study this problem since its formulation for closed waveguides is the same regardless of geometry. In this manner, randomness in the cross-section geometry can be contained in only the mesh. Furthermore, the triangular finite elements of the mesh can be projected onto a standardized triangle using Jacobian matrices, and hence every random instantiation of geometry can be modeled using the same standard mesh, preventing numerical discretization errors from polluting the statistical analysis. Other arbitrary waveguide cross-sections can also be analyzed by generating new meshes. In this chapter we consider the standard hollow rectangular waveguide since our goal is to verify the validity of the proposed stochastic analysis scheme, and we wish to compare our results against the well-known analytical solutions of these types of waveguides. The FEM formulation and stochastic collocation method will be presented in Section II. The results will be summarized and compared against analytical solutions and Monte Carlo results in Section III.

## II. Formulation

### A. Homogeneously-filled Waveguides

Electromagnetic waves can propagate in a PEC-walled waveguide if the wave shape corresponds to a propagating mode of the waveguide, and the frequency of the wave is above a certain cutoff threshold. An arbitrary wave can be described as a linear combination of many guided wave modes. Waves for each mode below its respective cutoff frequency will be attenuated and cannot propagate. The cutoff frequencies and dispersion characteristics of an infinitely long waveguide are determined by its cross-section. Propagating modes of a homogeneously-filled waveguide can be divided into two categories: Transverse Magnetic (TM) modes, where the magnetic field has only transverse components ($H_z = 0$), and Transverse Electric (TE) modes, where the electric field only has transverse components ($E_z = 0$). The two modes can be described by the Helmholtz wave equations:

$$\nabla_t^2 E_z + k_t^2 E_z = 0 \text{ in } \Omega \tag{1}$$

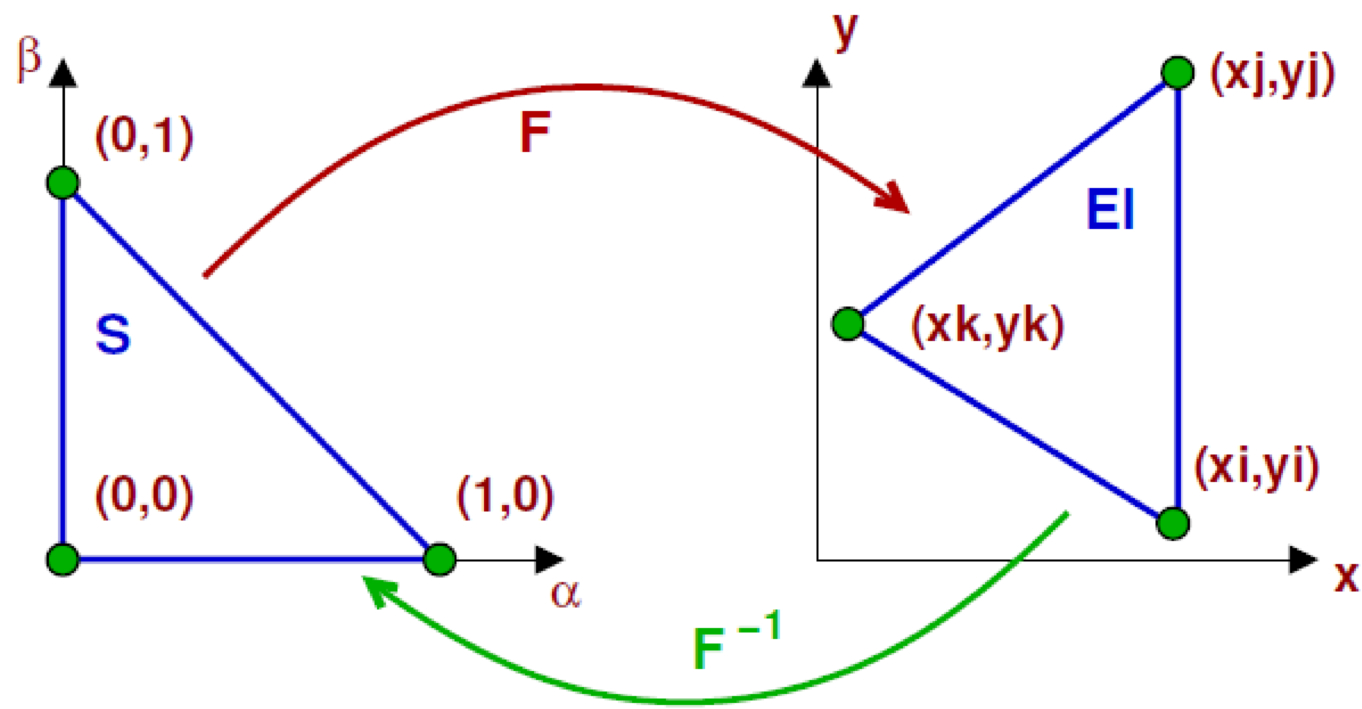


Fig. 1. Affine transformation $F$ that projects arbitrary finite element $e_i$ onto standardized triangle element $S$ [?].

$$\nabla_t^2 H_z + k_t^2 H_z = 0 \text{ in } \Omega \tag{2}$$

where $k_t$ is the transverse wavenumber, $\Omega$ is the cross-section of the waveguide, and $\nabla_t$ is the transverse del operator:

$$\nabla_t = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} \tag{3}$$

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{4}$$

The PEC wall of the waveguide can be described by a set of boundary conditions for the Helmholtz equations:

$$E_z = 0 \text{ on } \Gamma \tag{5}$$

$$\frac{\partial H_z}{\partial n} = 0 \text{ on } \Gamma \tag{6}$$

where $\Gamma$ is the PEC boundary of the waveguide.

The boundary value problem described by equations (1)-(6) has a countably infinite number of solutions. For each solution to $k_t$, we can find the corresponding propagation constant:

$$k_z^2 = k^2 - k_t^2, \quad k = \omega\sqrt{\mu\epsilon} \tag{7}$$

where $\omega$ is the angular frequency of the wave, and $\mu$ and $\epsilon$ are the permittivity and permeability of waveguide filling, respectively. For a hollow waveguide, we will use free-space values $\mu_0$ and $\epsilon_0$. We can easily see that in order for the propagation constant to have a real value, $k_t$ must be smaller than $k$. For these cases, the wave mode can propagate in the waveguide. If the propagation constant is imaginary, then the mode will be attenuated. Therefore the solutions $k_t$ are the cutoff wavenumbers of respective modes. The angular frequency at which $k = \omega\sqrt{\mu_0\epsilon_0} = k_t$ is the cutoff frequency. The propagation constants at each frequency can also be determined.

### B. *Finite Element Formulation*

The boundary value problem described in the section above can be converted into a system of algebraic equations. We first consider the TM case, where the only $\hat{z}$-direction field component is the electric field $E_z$. By applying Galerkin's method using nodal linear testing and basis functions to equation (1) and the boundary condition in equation (5), we obtain:

$$[A]\{E_z\} = k_c^2[B]\{E_z\} \tag{8}$$

$$A_{ij} = \int\int_\Omega \nabla_t N_i \cdot \nabla_t N_j \, d\Omega, \quad i,j = 1,2,...,N \tag{9}$$

$$B_{ij} = \int\int_\Omega N_i \cdot N_j \, d\Omega, \quad i,j = 1,2,...,N \tag{10}$$

where $N$ is the total number of nodes not on the Dirichlet boundary, and the basis functions $N_i$ are first order Lagrange polynomials with value 1 at node $i$ and 0 elsewhere.

To simplify the evaluation of the integrals, we can project each triangular finite element in the domain onto a unit right

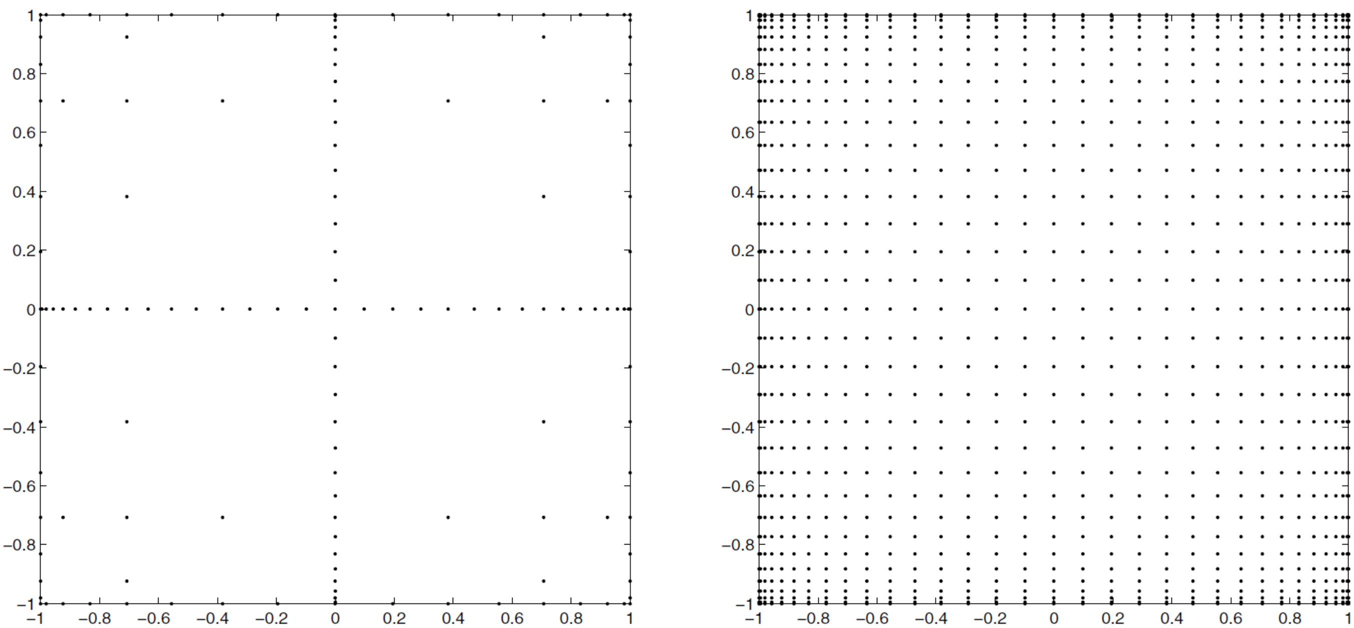

Fig. 2. Full tensor grid and Clenshaw-Curtis sparse grid in 2 dimensional random space. The sparse grid is plotted for level $k = 5$ and has 145 nodes. The tensor grid has 1089 nodes. [**?**]

triangle (Figure 1) through an affine transformation $F$ [**?**]. The integrals can then be performed over the area of the right triangle:

$$A_e = \int\int_E \nabla_t N_r \cdot \nabla_t N_s \, dE = (J^{-T}\nabla\lambda_r)^T (J^{-T}\nabla\lambda_s)\,|J| \int_S dS \tag{11}$$

$$B_e = \int\int_E N_r \cdot N_s \, dE = |J| \int_S \lambda_r \cdot \lambda_s \, dS \tag{12}$$

$$J = \begin{bmatrix} (x_j - x_i) & (x_k - x_i) \\ (y_j - y_i) & (y_k - y_i) \end{bmatrix} \tag{13}$$

$$\nabla\lambda = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \tag{14}$$

$$\int_S dS = \frac{1}{2}, \quad \int_S \lambda_r \cdot \lambda_s \, dS = \begin{bmatrix} 1/12 & 1/24 & 1/24 \\ 1/24 & 1/12 & 1/24 \\ 1/24 & 1/24 & 1/12 \end{bmatrix} \tag{15}$$

where $E$ is the domain of the triangular element, $S$ is the domain of the reference triangle, $J$ is the Jacobian of the affine transformation, and $\lambda$ is the basis function over the reference triangle. $A_e$ and $B_e$ will be $3 \times 3$ matrices, and each entry can be assembled into the global $A$ and $B$ matrices based on the nodal connectivity array. For each node $i$ on the Dirichlet boundaries, $A_{ii}$ should be set to 1, while all other entries in $i$th row and column should be set to 0.

Likewise, for TE cases, the boundary value problem can be formulated as:

$$[A]\{H_z\} = k_c^2 [B]\{H_z\} \tag{16}$$

Unlike in the TM case, the PEC walls of the waveguide act as Neumann boundaries for the magnetic field; therefore, the $A$ and $B$ matrices need to contain entries for all nodes in the mesh, including those on the boundaries.

The algebraic system of equations (8) and (16) is in the form of generalized eigenvalue problems. These equations can be solved using readily available eigensolvers. The resulting eigenvalues represent squares of the cutoff frequencies of various propagating modes, and the corresponding eigenvectors represent field distributions in the waveguide cross-section for the corresponding mode. The field distribution of each mode can be visualized by plotting the eigenvectors. Since the matrices $A$ and $B$ have very few non-zero entries, the computational efficiency of the eigendecomposition process can be improved drastically if these matrices are stored and processed as sparse matrices. This is critical for dense meshes since the computational complexity of a non-sparse eigendecomposition process is $O(n^3)$ while sparse algorithms can achieve $O(n \log n)$ performance.

### C. Stochastic Computations

When there are uncertainties in the waveguide cross-section, we expect the cutoff frequencies and propagation constants of each mode to also be uncertain. Our goal now is to find the probability density function (PDF) of cutoff frequencies and propagation constants for each mode, using PDFs of the geometries as inputs. The classical approach for solving this type of problem is by the Monte Carlo (MC) method, where a large number of input samples are generated conforming to the PDFs of the geometry, and the solution computed for each instantiation. PDFs of the cutoff frequencies, etc., can then be obtained statistically from the solutions. A crucial limitation to MC is its slow speed and convergence. According to the Central Limit Theorem:

$$\frac{S_n - n\mu}{\sqrt{n}} \xrightarrow{d.} \mathcal{N}(0, \sigma^2) \text{ as } n \to \infty \tag{17}$$

which means that the expectation converges at the rate of $\frac{1}{\sqrt{N}}$ for $N$ numbers of samples. Therefore a large sample size is needed to achieve convergence in distribution. For each sample, a full FEM analysis needs to be performed.

A much more efficient method for calculating the desired PDFs is through stochastic collocation. Stochastic collocation is based on the idea that random variables can be expanded as the sum of orthogonal polynomials [**?**]:

$$X(\boldsymbol{\xi}) = \sum_{k=0}^{P} X_k \cdot \phi_k(\boldsymbol{\xi}) \tag{18}$$

By the orthogonality principle, for continuous random variable $X$ with PDF $\rho(\xi)$,

$$\begin{aligned} \mathbb{E}[\phi_m(X)\phi_n(X)] &= \int \phi_m(\boldsymbol{\xi})\phi_n(\boldsymbol{\xi})\rho(\boldsymbol{\xi})d\boldsymbol{\xi} \\ &= \begin{cases} 0 \text{ for } m \neq n \\ \mathbb{E}[\phi_m^2] \text{ for } m = n \end{cases} \end{aligned} \tag{19}$$

Similar to FEM method for electromagnetics, we can choose first-order Lagrange polynomials as the basis functions for the expansion in (18) [**?**]. This is a linear interpolation approach to stochastic collocation:

$$w(Z) = \sum_{j=1}^{M} u(Z^{(j)})N_j(Z) \tag{20}$$

where $Z$ is the input random vector, and the function is evaluated at a set of nodes $\Theta_M$ in the input random space. The weights $u(Z^{(j)})$ are calculated at each node and used to interpolate the response surface in random space. The PDF can then be extracted by performing MC sampling on the interpolated response surface.

The most straightforward nodal set for interpolation is a full tensor product set in the $d$-dimensional random space:

$$\Theta_M = \Theta_1^{m_1} \times ... \times \Theta_1^{m_d} \tag{21}$$

where each random dimension is discretized into $m$ points. The total number of nodes is $M = m_1 \times ... \times m_d$. We can see that the total number of nodes grows exponentially with the number of random dimensions, a phenomenon known as the "curse of dimensionality". A more efficient alternative approach is to use Smolyak sparse grids for interpolation:

$$\Theta_M = \bigcup_{N-d+1 \leq |i| \leq N} (\Theta_1^{i_1} \times ... \times \Theta_1^{i_d}) \tag{22}$$

where $N \geq d$ is an integer denoting the level of the construction. The type of sparse grid we use in this chapter is Clenshaw-Curtis nodes, which are extrema of Chebyshev polynomials defined by:

$$Z_i^{(j)} = -\cos\frac{\pi(j-1)}{m_i^k - 1}, \quad j = 1, ..., m_i^k \tag{23}$$

where $k = N - d$ is the level of construction, and the total number of nodes is $m_i^k = 2^{k-1} + 1$. By using this sparse grid, the response surface can be approximated with much fewer number of nodes. Figure 2 shows a full tensor grid and a Clenshaw-Curtis sparse grid over the same 2-dimensional space.

## III. Results

### A. Problem Set-up

We study a hollow rectangular waveguide with PEC walls and a nominally 2 meters $\times$ 1 meter cross-section. The analytical solution for the cutoff wavenumber is:

$$k_c = \sqrt{\left(\frac{m\pi}{A}\right)^2 + \left(\frac{n\pi}{B}\right)^2} \tag{24}$$

for $\text{TM}_{mn}$ or $\text{TE}_{mn}$ modes of the waveguide. We denote the cross-section dimensions $A$ and $B$ using capital letters since they are random variables:

$$A \sim \mathcal{N}(2, \sqrt{0.06}), \quad B \sim \mathcal{N}(1, \sqrt{0.03}) \tag{25}$$

and we can normalize these random variables:

$$\begin{aligned} A &= 2(1 + 0.03\ \xi_1), \\ B &= 1(1 + 0.03\ \xi_2), \\ \xi_1, \xi_2 &\sim \mathcal{N}(0,1)\ i.i.d. \end{aligned} \tag{26}$$

such that samples of the Gaussian random variables can be generated readily using existing random number generator in packages like MATLAB.

We generate a triangular mesh on the mean-dimension rectangle, shown in Figure 3. The density is chosen so that higher order modes can be calculated accurately. In general, the element size should be less than 1/20 of the wavelength. By utilizing the Jacobian formulation in (13), we can use the same mesh for all random variations of the problem by simply scaling the Jacobian with $\xi_1$ and $\xi_2$. In this sense, the stochastics is formulated into the FEM problem by random stretchings of the mesh.

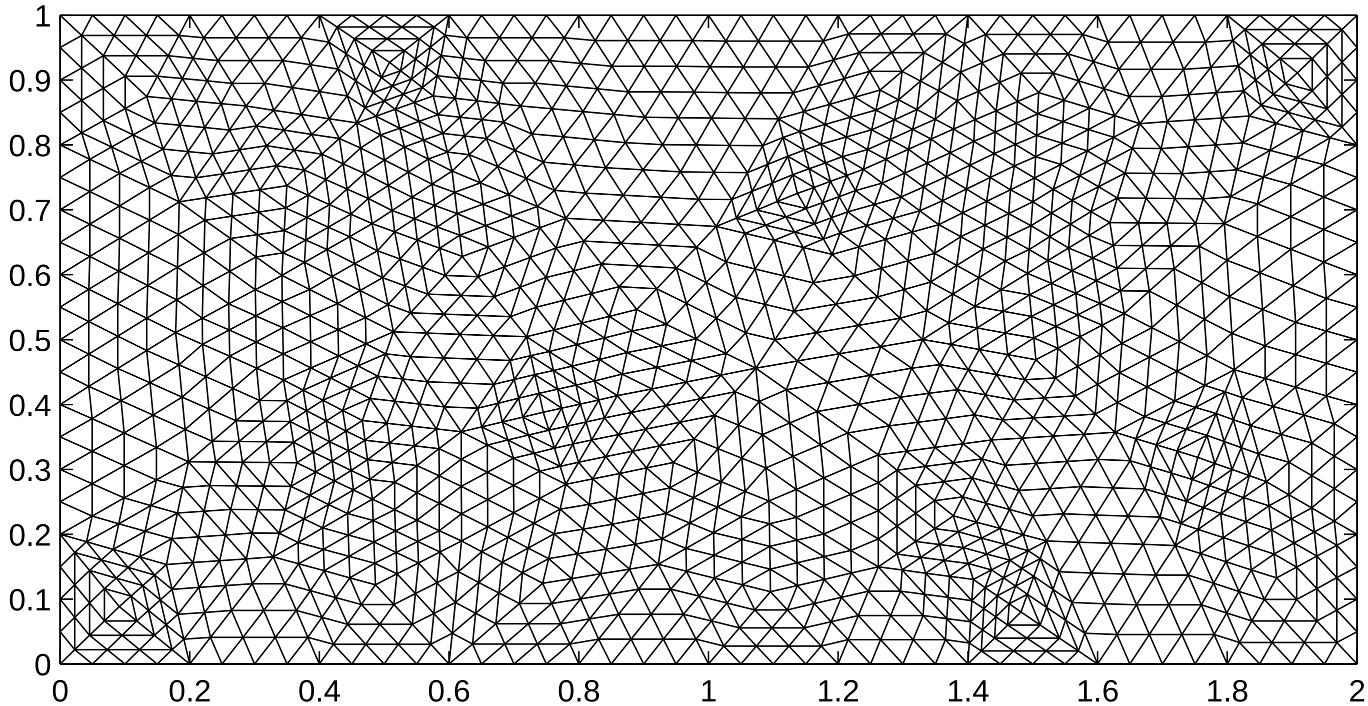


Fig. 3. Finite element mesh of a 2m × 1m rectangular waveguide cross-section. There are 2688 triangular elements and 1405 nodes.

### B. *Finite Element Solutions Of Waveguide*

By solving the eigenvalue problems formulated in (8) and (16), we obtain a series of eigenvalues that correspond to the square of the cutoff wavenumbers of various propagating modes. These values are checked against the analytical values obtained from (24) to ensure that the FEM program is functioning correctly. A comparison is shown in Table I. The corresponding eigenvectors represent the $E_z$ and $H_z$ distribution on the waveguide cross-section at each node. By interpolating the values between nodes, plots of the field distribution for various propagating modes can be generated. The plots in Figure 5 are sorted by cutoff frequency.

The dispersion diagram of the modes can be calculated from the solutions. This is shown in Figure 4 for the first seven modes. Note that below cutoff, the wave is attenuated and does not propagate. There are three degenerate pairs (TE01 & TE20, TM11 & TE11, TM21 & TE21).

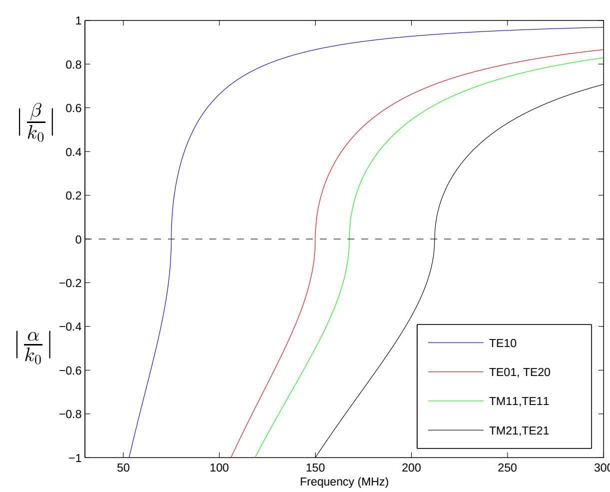


Fig. 4. Dispersion diagram for the first 7 modes in a 2m × 1m rectangular waveguide calculated using FEM. There are 3 degenerate pairs.

### C. *Waveguide Solutions With Uncertainties*

With the validity of the FEM program verified, we study the waveguide with stochastic dimensions. We assume that each dimension will vary around 10 percent, which correspond to a standard deviation of approximately 3 percent. The PDF of the dimensions are modeled using the random variables described in (26). In order to obtain the PDF of $k_c$, we use the sparse grid stochastic collocation method outlined in (20) and (23). We also verify the results using Monte Carlo method with $N = 5000$ and analytical method (24).

We can see in Figure 6 that the results show good agreement. On a 2.3 GHz Intel Xeon system, the stochastic collocation method takes about 70 seconds while the MC method took over 80 minutes to complete. The statistics of mode dispersion can also be easily calculated from stochastic collocation results. Figure 7 shows the dispersion diagram for the first three modes of the waveguide, with the mean and $\pm 3\sigma$ values. These results gives us insight into the range of propagation constants we can expect when the geometry is uncertain.

TABLE I

CUTOFF ANALYSIS OF A 2M × 1M RECTANGULAR WAVEGUIDE: ANALYTICAL SOLUTION VS. FEM SOLUTION FOR THE FIRST 10 MODES.

| Mode | $k_c^2$ (Analytical) [$\mathrm{m}^{-2}$] | $k_c^2$ (FEM) [$\mathrm{m}^{-2}$] | $f_c$ (Analytical) [MHz] | $f_c$ (FEM) [MHz] | % Error |
|---|---|---|---|---|---|
| TE10 | 2.4674 | 2.4676 | 74.9481 | 74.9513 | 0.0043 |
| TE01 | 9.8696 | 9.8724 | 149.8962 | 149.9175 | 0.0142 |
| TE20 | 9.8696 | 9.8734 | 149.8962 | 149.9251 | 0.0193 |
| TM11 | 12.3370 | 12.3413 | 167.5890 | 167.6186 | 0.0176 |
| TE11 | 12.3370 | 12.3416 | 167.5890 | 167.6206 | 0.0188 |
| TM21 | 19.7392 | 19.7514 | 211.9852 | 212.0510 | 0.0310 |
| TE21 | 19.7392 | 19.7516 | 211.9852 | 212.0521 | 0.0315 |
| TE30 | 22.2066 | 22.2244 | 224.8443 | 224.9345 | 0.0401 |
| TM31 | 32.0762 | 32.1089 | 270.2292 | 270.3669 | 0.0509 |
| TE31 | 32.0762 | 32.1108 | 270.2292 | 270.3749 | 0.0539 |

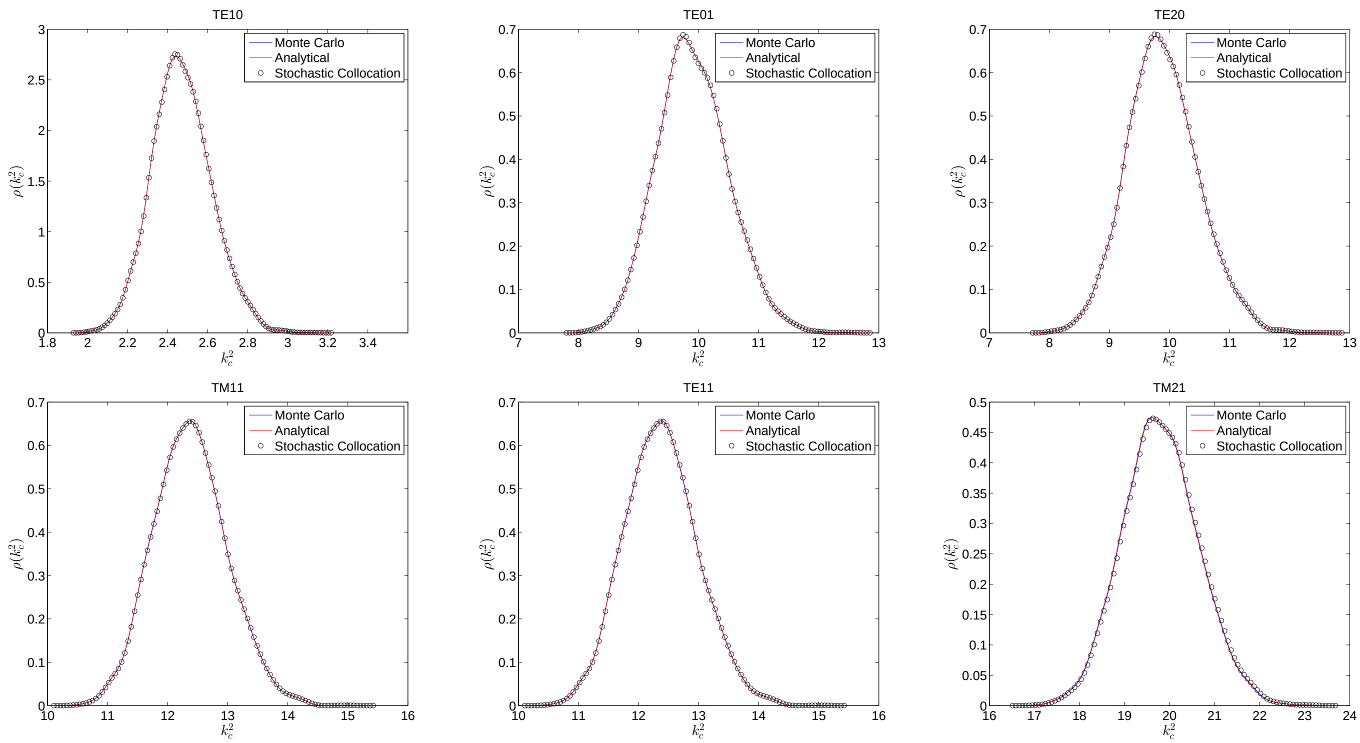


Fig. 6. Probability density functions (PDFs) of $k_c^2$ for TE10, TE01, TE20, TM11, TE11, and TM21 modes in a rectangular waveguide with geometric uncertainty. Note that the degenerate pair TE01 and TE20 have different probability distributions.

## IV. REMARKS

The Finite Element Method is a highly robust numerical scheme for solving boundary value problems. This chapter studied an example with regular geometry, but any irregular and complex geometry can be computed using the same formulation by generating a mesh for it. Due to FEM's easily adaptivity to changing geometries, it is the ideal method to study stochastic problems where geometry is random. By using affine transformation of the finite elements, we are able to model the random variables into the Jacobian matrix, thus eliminating the need to re-mesh. This same technique can be easily applied to other problems where the size is random.

The stochastic collocation method used for this chapter allows for extremely fast computation of random variables without the need for lengthy Monte Carlo type simulations. The benefits and accuracy of the method are demonstrated with our example. We can also see that uncertainties in electromagnetic problems can significantly affect the results. In many engineering applications, knowledge of the distribution of results is more important than having the most accurate result for a deterministic problem. Numerical methods such as stochastic collocation will give engineers powerful tools in modeling electromagnetic problems with random inputs.

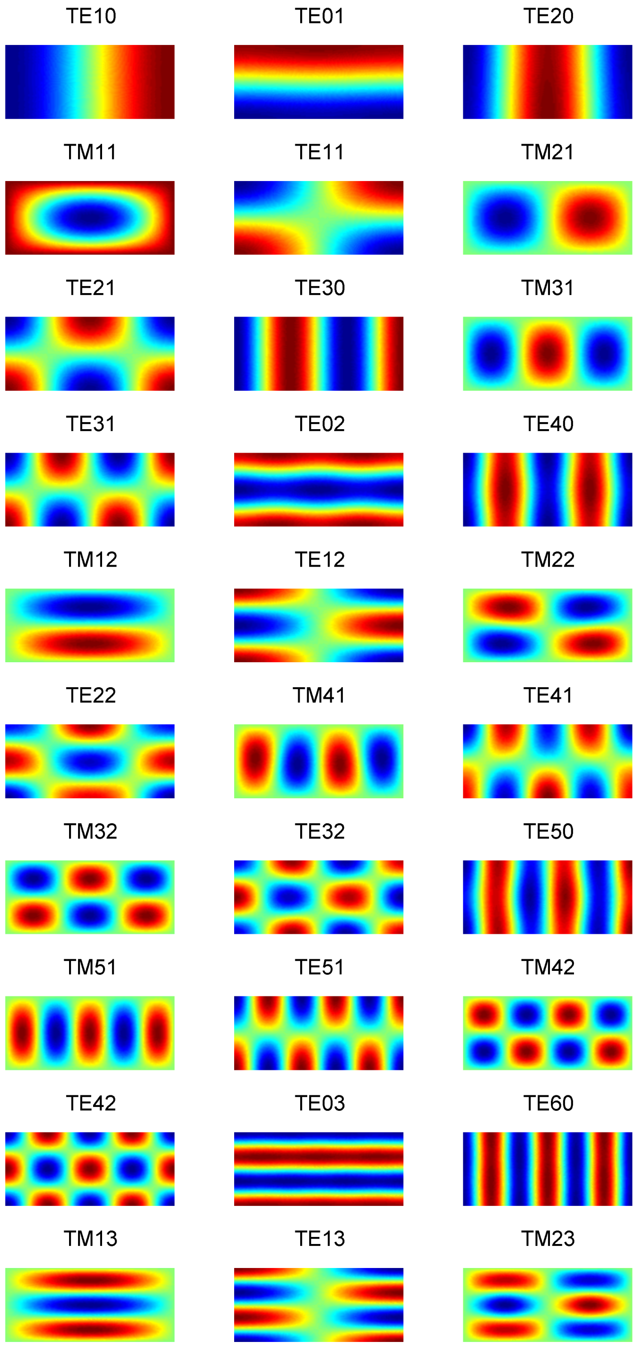


Fig. 5. Transverse field distribution of the first 30 modes in a 2:1 rectangular waveguide.

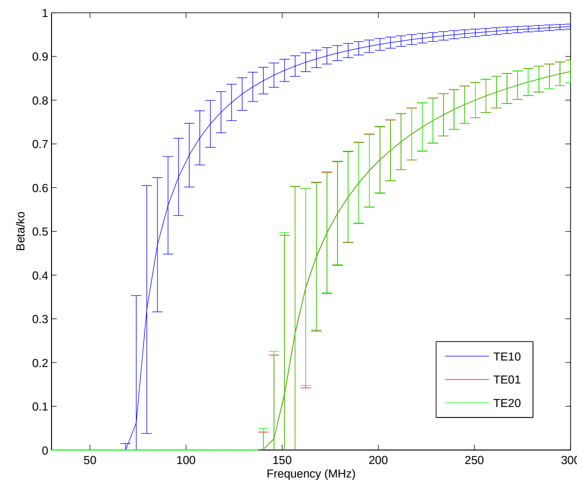


Fig. 7. Stochastic dispersion diagram for the first 3 propagating modes of the waveguide with geometric uncertainties. Ranges represent $\pm 3\sigma$.

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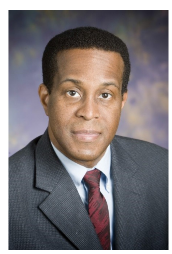

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